

# DISTORTION OF QUASICONFORMAL MAPPINGS WITH IDENTITY BOUNDARY VALUES

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**ABSTRACT.** Teichmüller's classical mapping problem for plane domains concerns finding a lower bound for the maximal dilatation of a quasiconformal homeomorphism which holds the boundary pointwise fixed, maps the domain onto itself, and maps a given point of the domain to another given point of the domain. For a domain  $D \subset \mathbb{R}^n$ ,  $n \geq 2$ , we consider the class of all  $K$ -quasiconformal maps of  $D$  onto itself with identity boundary values and Teichmüller's problem in this context. Given a map  $f$  of this class and a point  $x \in D$ , we show that the maximal dilatation of  $f$  has a lower bound in terms of the distance of  $x$  and  $f(x)$ . We improve recent results for the unit ball and consider this problem in other more general domains. For instance, convex domains, bounded domains and domains with uniformly perfect boundaries are studied.

**KEYWORDS.** quasiconformal mappings, identity boundary values, distortion theorems

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## 1. INTRODUCTION

Teichmüller's classical problem for quasiconformal automorphisms of plane domains has been studied by G.J. Martin in [M1] and most recently in [M2]. Of these two papers the second one concentrates on this problem from the point of view of the mean dilatation. For the case of the unit ball of  $\mathbb{R}^n$ ,  $n \geq 3$ , an asymptotically sharp bound when the maximal dilatation  $K \rightarrow 1$  was given by V. Manojlović and M. Vuorinen [MV]. Continuing the work of [MV], we study the case of subdomains of  $\mathbb{R}^n$  more general than the unit ball. For basic information on quasiconformal maps in  $\mathbb{R}^n$  we refer the reader to [V1].

Let  $D$  be a proper subdomain of  $\mathbb{R}^n$  ( $n \geq 2$ ), and let

$$\text{Id}_K(\partial D) = \{f : \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ is } K\text{-quasiconformal} : f(x) = x, \forall x \in \mathbb{R}^n \setminus D\}.$$

In his classical work [T] O. Teichmüller studied the class  $\text{Id}_K(\partial D)$  with  $D = \mathbb{R}^2 \setminus \{0, e_1\}$  and proved that the following sharp inequality

$$s_D(x, f(x)) \leq \log K$$

holds for all  $x \in D$ , where  $s_D$  is the hyperbolic metric of  $D = \mathbb{R}^2 \setminus \{0, e_1\}$ . This result may be regarded as a stability result since it says that  $f(x)$  is contained in the closure of the hyperbolic ball  $B_{s_D}(x, \log K)$  centered at the point  $x$  and with the radius  $\log K$ . In particular, the radius tends to 0 as  $K \rightarrow 1$ .

J. Krzyż [K] considered the same problem for the case of the unit disk, and G. D. Anderson and M. K. Vamanamurthy [AV] found a counterpart for Krzyż's result

in the case of the unit ball in  $\mathbb{R}^n$ ,  $n \geq 3$ , under an additional symmetry hypothesis. Very recently, V. Manojlović and M. Vuorinen [MV] removed the extra symmetry hypothesis and proved the following Theorem 1.1.

As in [Vu3, p. 97 (7.44), p. 138, Theorem 11.2], we denote by  $\varphi_{K,n} : [0, 1] \rightarrow [0, 1]$ ,  $K > 0$ , the special function connected with the Schwarz lemma. Its precise definition of  $\varphi_{K,n}$  is given in (2.3). What is important is that it is an increasing homeomorphism with  $\varphi_{K,n}(r) \rightarrow r$  as  $K \rightarrow 1$ .

**1.1. Theorem.** [MV, Theorem 1.9] *If  $f \in \text{Id}_K(\partial\mathbb{B}^n)$ , then, for all  $x \in \mathbb{B}^n$ ,*

$$\rho_{\mathbb{B}^n}(x, f(x)) \leq \log \frac{1-a}{a}, \quad a = \varphi_{1/K,n}(1/\sqrt{2})^2,$$

where  $\rho_{\mathbb{B}^n}$  is the hyperbolic metric defined in (2.1).

Theorem 1.1 shows that the mapping  $f$  uniformly tends to the identity mapping when the maximal dilatation goes to 1.

In this paper we will first improve Manojlović and Vuorinen's result as follows.

**1.2. Theorem.** *If  $f \in \text{Id}_K(\partial\mathbb{B}^n)$ , then, for all  $x \in \mathbb{B}^n$ ,*

$$\rho_{\mathbb{B}^n}(x, f(x)) \leq \log \frac{2\varphi_{K,n}(1/3)}{1 - \varphi_{K,n}(1/3)}.$$

A comparison shows that Theorem 1.2 yields a better bound than Theorem 1.1 (see Figure 2) when  $n = 2$ . Both Theorem 1.1 and Theorem 1.2 imply that if  $f(x) \neq x$  for some  $x \in \mathbb{B}^n$  then  $K > 1$ .

In the case of uniform domain with connected boundary, M. Vuorinen [Vu1] established

$$(1.3) \quad K \geq c_1(n, D) k_D(x, f(x))^n$$

whenever the quasihyperbolic distance  $k_D(x, f(x))$  exceeds a bound depending only on  $n$  and  $D$ . Here  $c_1(n, D)$  is a positive constant depending only on  $n$  and  $D$ . As pointed out in [Vu1], it is not true for  $n \geq 3$  that  $k_D(x, f(x)) > 0$  implies  $K > 1$ . Indeed, let  $X = \{(x, 0, 0) : x \in \mathbb{R}\}$  be the  $x_1$ -axis, let  $D = \mathbb{R}^3 \setminus X$ , and let  $f : D \rightarrow D$  be a rotation around the  $x_1$ -axis with  $f(x) = (0, -1, 0)$ ,  $x = (0, 1, 0)$ . Then  $f$  is conformal, i.e.  $K = 1$ ,  $f$  keeps the  $x_1$ -axis  $X = \partial D$  pointwise fixed, and  $D$  is a uniform domain with connected boundary  $X$  and  $k_D(x, f(x)) = \pi$ . Clearly, for this domain  $c_1(3, D) \leq 1/\pi^3$ .

For convex domains, however, we show that the requirement  $f(x) \neq x$  for some  $x \in D$  implies the maximal dilatation of  $f$  to be greater than 1. This kind of behavior also holds for bounded domains as Theorem 3.3 shows.

**1.4. Theorem.** *Let  $D \subsetneq \mathbb{R}^n$  be a convex domain and  $f \in \text{Id}_K(\partial D)$ . Then for all  $x \in D$*

$$j_D(x, f(x)) \leq \log \left( 1 + \sqrt{c_2(n, K)^2 - 1} \right) \leq \log \left( 1 + \frac{\sqrt{1-2a}}{a} \right)$$

where  $c_2(n, K) = \min\{\eta_{K,n}(1), \eta_{1/K,n}(1)^{-1}\}$  with  $c_2(n, K) \rightarrow 1$  as  $K \rightarrow 1$ , and  $a = \varphi_{1/K,n}(1/\sqrt{2})^2 \rightarrow 1/2$  as  $K \rightarrow 1$ . Here  $j_D$  is the distance ratio metric in  $D$ .

For  $K$  close to 1, the inequality of Theorem 1.4 can be simplified further.

**1.5. Theorem.** *Let  $D \subsetneq \mathbb{R}^n$  be a convex domain and*

$$K_n = \left(1 + \frac{\log 3}{2(n-1) + \log 8}\right)^{n-1} \in [K_2, \sqrt{3}), \quad K_2 \approx 1.2693,$$

*and let  $K \in (1, K_n]$  and  $f \in \text{Id}_K(\partial D)$ . Then for all  $x \in D$*

$$j_D(x, f(x)) \leq 4\sqrt{K-1}.$$

The proof of the inequality (1.3) makes use of the classical Väisälä lower bound for the modulus of the family of curves joining continua [V1, Theorem 10.12]. Aseev's theorem [As, Theorem 3] (see Lemma 2.7 below) provides a counterpart of this result with continua replaced with uniformly perfect sets. In this way we can prove that (1.3) also holds for the case of uniform domains with uniformly perfect boundary.

**1.6. Theorem.** *Let  $D \subsetneq \mathbb{R}^n$  be a uniform domain with uniformly perfect boundary and  $x \in D$ , and let  $f \in \text{Id}_K(\partial D)$ . Then there exists a positive constant  $c_3(n, D)$  depending only on  $n$  and the constants of uniformity and uniform perfectness of the domain  $D$  such that for all  $x \in D$*

$$K \geq c_3(n, D)k_D(x, f(x))^n.$$

The Hölder continuity of quasiconformal self mappings of the unit ball with the origin fixed is an important topic which was first studied by Ahlfors [Ah] when the dimension  $n = 2$ . Refining Ahlfors' result, A. Mori proved that a  $K$ -quasiconformal mapping  $f : \mathbb{B}^2 \rightarrow \mathbb{B}^2$ ,  $f(0) = 0$ , satisfies for all  $x, y \in \mathbb{B}^2$  the inequality

$$|f(x) - f(y)| \leq M|x - y|^{1/K}$$

with the best possible constant  $M = 16$  independent of  $K$  and the sharp exponent  $1/K$  [LV]. Later on, it was conjectured that here 16 can be replaced by  $16^{1-1/K}$ . This conjecture, sometimes referred to as the *Mori's conjecture for planar quasiconformal maps of the unit disk*, is a well-known open problem and it has been studied by many people. For the higher dimensional case  $n \geq 3$  an asymptotically sharp constant, i.e. a constant tending to 1 when  $K \rightarrow 1$ , was proved for the first time by Fehlmann and Vuorinen [FV]. Very recently, Bhayo and Vuorinen [BV] improved the previous results by using a refined inequality for the Teichmüller function and introducing an additional parameter which was chosen in an optimal way. Many authors have studied these questions. For the detailed history of the Hölder continuity of quasiconformal mappings, the readers are referred to the bibliographies of [MRV], [FV], [Vu3] and [BV]. We will consider this problem and improve the constant for the class of quasiconformal mappings of the unit ball with identity boundary values. Note that in this case it is not required that the origin be fixed by the mapping.

**1.7. Theorem.** *If  $f \in \text{Id}_K(\partial \mathbb{B}^n)$ , then for all  $x, y \in \mathbb{B}^n$*

$$|f(x) - f(y)| \leq M_1(n, K)|x - y|^\alpha, \quad \alpha = K^{1/(1-n)}$$

*where  $M_1(n, K) = \lambda_n^{1-\alpha} C(\alpha)$  and  $C(\alpha) = 2^{1-\alpha} \alpha^{-\alpha/2} (1-\alpha)^{(\alpha-1)/2}$ , with  $M_1(n, K) \rightarrow 1$  when  $K \rightarrow 1$ , and  $\lambda_n \in [4, 2e^{n-1})$  is the Grötzsch ring constant.*

## 2. NOTATION AND PRELIMINARY RESULTS

In this section we shall follow the standard notation and terminology for  $K$ -quasi-conformal mappings in the Euclidean  $n$ -space  $\mathbb{R}^n$ , see e.g. [AVV2], [V1] and [Vu3].

The *hyperbolic metric*  $\rho_{\mathbb{B}^n}(x, y)$  on  $\mathbb{B}^n$  is defined by

$$(2.1) \quad \tanh^2 \frac{\rho_{\mathbb{B}^n}(x, y)}{2} = \frac{|x - y|^2}{|x - y|^2 + (1 - |x|^2)(1 - |y|^2)}.$$

A simple argument shows that we have

$$|x - y| \leq 2 \tanh \frac{\rho_{\mathbb{B}^n}(x, y)}{4}$$

for all  $x, y \in \mathbb{B}^n$  with equality for  $x = -y$  (see [Vu3, (2.27)]).

Let  $D \subsetneq \mathbb{R}^n$  be a domain. The *quasihyperbolic metric*  $k_D$  is defined by [GP]

$$k_D(x, y) = \inf_{\gamma \in \Gamma} \int_{\gamma} \frac{1}{d(z)} |dz|, \quad x, y \in D,$$

where  $\Gamma$  is the family of all rectifiable curves in  $D$  joining  $x$  and  $y$ , and  $d(z) = d(z, \partial D)$  is the Euclidean distance between  $z$  and the boundary of  $D$ . The *distance-ratio metric* or *j-metric* is defined as [GP, Vu2]

$$(2.2) \quad j_D(x, y) = \log \left( 1 + \frac{|x - y|}{\min\{d(x), d(y)\}} \right), \quad x, y \in D.$$

It is well known that [GP, Lemma 2.1], [Vu3, (3.4)]

$$j_D(x, y) \leq k_D(x, y)$$

for all domains  $D \subsetneq \mathbb{R}^n$  and  $x, y \in D$ .

A domain  $D$  in  $\mathbb{R}^n$ ,  $D \neq \mathbb{R}^n$ , is called *uniform*, if there exists a number  $U = U(D) \geq 1$  such that  $k_D(x, y) \leq U j_D(x, y)$  for all  $x, y \in D$ . Uniform domains were introduced by Martio and Sarvas [MS]. Presently, there are several equivalent definitions of uniform domains, see, for instance, Väisälä [V2]. The above definition, which is most convenient for the sequel, is adopted from Gehring and Osgood [GO] and Vuorinen [Vu2].

It is well known [AVV2, Lemma 7.56] that the unit ball  $\mathbb{B}^n$  is a uniform domain with the constant  $U = 2$ .

Given  $E, F, G \subset \mathbb{R}^n$  we use the notation  $\Delta(E, F; G)$  for the family of all curves that join the sets  $E$  and  $F$  in  $G$  and  $M(\Delta(E, F; G))$  for its modulus. If  $G = \mathbb{R}^n$ , we may omit  $G$  and simply denote  $\Delta(E, F; G)$  by  $\Delta(E, F)$ . For a ring domain  $R(C_0, C_1)$  with complementary components  $C_0$  and  $C_1$ , we define the modulus of  $R(C_0, C_1)$  by

$$\text{mod} R(C_0, C_1) = \left( \frac{\omega_{n-1}}{M(\Delta(C_0, C_1))} \right)^{1/(n-1)},$$

where  $\omega_{n-1}$  is the surface area of the unit sphere in  $\mathbb{R}^n$ .

The *Grötzsch ring domain*  $R_{G,n}(s)$ ,  $s > 1$ , and the *Teichmüller ring domain*  $R_{T,n}(t)$ ,  $t > 0$ , are doubly connected domains with complementary components  $(\mathbb{B}^n, [se_1, \infty))$  and  $([-e_1, 0], [te_1, \infty))$ , respectively. For their capacities we write

$$\begin{cases} \gamma_n(s) = \text{cap} R_{G,n}(s) = M(\Delta(\overline{\mathbb{B}^n}, [se_1, \infty))), \\ \tau_n(t) = \text{cap} R_{T,n}(t) = M(\Delta([-e_1, 0], [te_1, \infty))). \end{cases}$$

These functions are related by the functional identity

$$\gamma_n(s) = 2^{n-1} \tau_n(s^2 - 1).$$

For  $K > 0$  we define an increasing homeomorphism  $\varphi_{K,n} : [0, 1] \rightarrow [0, 1]$  with  $\varphi_{K,n}(0) = 0$ ,  $\varphi_{K,n}(1) = 1$  and

$$(2.3) \quad \varphi_{K,n}(r) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/r))}, \quad 0 < r < 1.$$

The following important estimates are well known [Vu3]

$$(2.4) \quad r^\alpha \leq \varphi_{K,n}(r) \leq \lambda_n^{1-\alpha} r^\alpha \leq 2^{1-1/K} K r^\alpha, \quad \alpha = K^{1/(1-n)},$$

$$(2.5) \quad 2^{1-K} K^{-K} r^\beta \leq \lambda_n^{1-\beta} r^\beta \leq \varphi_{1/K,n}(r) \leq r^\beta, \quad \beta = 1/\alpha,$$

where  $K \geq 1$ ,  $r \in (0, 1)$ , and the constant  $\lambda_n \in [4, 2e^{n-1})$  is the so-called *Grötzsch ring constant*. In particular,  $\lambda_2 = 4$ .

For  $n \geq 2$ ,  $t \in (0, \infty)$ ,  $K > 0$ , we denote

$$(2.6) \quad \eta_{K,n}(t) = \tau_n^{-1} \left( \frac{1}{K} \tau_n(t) \right) = \frac{1 - \varphi_{1/K,n}(1/\sqrt{1+t})^2}{\varphi_{1/K,n}(1/\sqrt{1+t})^2}.$$

Let  $\alpha > 0$  and assume that  $D \subset \overline{\mathbb{R}^n}$  is a closed set containing at least two points. Then  $D$  is *s-uniformly perfect* if there is no ring domain separating  $D$  with the modulus greater than  $s$ .  $D$  is *uniformly perfect* if it is *s-uniformly perfect* for some  $s > 0$ . Uniformly perfectness is a useful tool in many topics of geometric function theory. See [S] for a survey of this topic. The following lemma is an analog of Väisälä's lemma [V1, Theorem 10.12] with continua replaced by uniformly perfect sets.

**2.7. Lemma.** [As, Theorem 3] *Suppose that  $s > 0$  and that  $s$ -uniformly perfect sets  $E_0$  and  $E_1$  meets each component of the complement of the spherical ring  $D = \{x : r_1 < |x - x_0| < r_2\} \subset \mathbb{R}^n$  with the following relation between the radii*

$$r_2/r_1 > 1 + 2e^s.$$

*Then*

$$\text{cap}(E_0, E_1; D) \geq C \log \frac{r_2}{r_1},$$

*where the constant  $C > 0$  depends only on  $s$  and the dimension  $n$  of the space.*

Note that, from the proof of this lemma, the result obviously holds if one of the two sets  $E_0$  and  $E_1$  is a continuum.

### 3. PROOFS OF MAIN RESULTS

**3.1. Proof of Theorem 1.2.** For arbitrarily given  $x \in \mathbb{B}^n$  let  $T_x$  be the Möbius transformation of  $\overline{\mathbb{R}^n}$  with  $T_x(\mathbb{B}^n) = \mathbb{B}^n$  and  $T_x(x) = 0$  [B]. Define  $g : \overline{\mathbb{R}^n} \rightarrow \overline{\mathbb{R}^n}$  by setting  $g(z) = T_x \circ f \circ T_x^{-1}(z)$  for  $z \in \mathbb{B}^n$  and  $g(z) = z$  for  $z \in \overline{\mathbb{R}^n} \setminus \mathbb{B}^n$ . Then  $g \in \text{Id}_K(\partial \mathbb{B}^n)$  with  $g(0) = T_x(f(x))$ . Since the hyperbolic metric  $\rho_{\mathbb{B}^n}$  is preserved under Möbius transformations of  $\mathbb{B}^n$  onto itself, we have that for  $x \in \mathbb{B}^n$

$$(3.2) \quad \rho_{\mathbb{B}^n}(x, f(x)) = \rho_{\mathbb{B}^n}(0, g(0)).$$

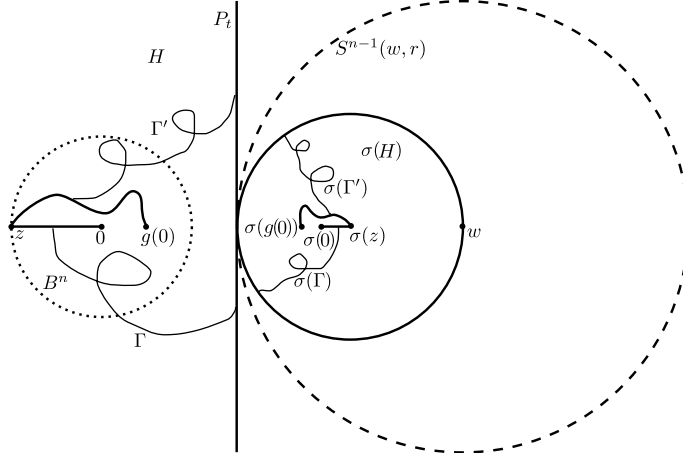


FIGURE 1. The proof of Theorem 1.2 visualized.

Choose  $z \in \partial \mathbb{B}^n$  such that the origin is contained in the segment  $[g(0), z]$ . For  $t \geq 0$  let  $P_t = P(g(0), (1+t)|g(0)|) = \{x \in \mathbb{R}^n : x \cdot g(0) = (1+t)|g(0)|\}$  be the hyperplane in  $\mathbb{R}^n$  perpendicular to the vector  $g(0)$ , at distance  $1+t$  from the origin [B], and the half space  $H$  be the component of  $\mathbb{R}^n \setminus P_t$  which contains the origin. Let  $\sigma$  be the inversion in the sphere  $S^{n-1}(w, r)$  where  $w = (3+2t)g(0)/|g(0)|$  and  $r = 2+t$ , then we have  $\sigma(z) = (4+3t)g(0)/(2|g(0)|)$  and  $\sigma(H) = \mathbb{B}^n(\sigma(z), r/2)$ . It is easy to see that

$$|\sigma(z) - \sigma(0)| = \frac{2+t}{6+4t}$$

and

$$|\sigma(z) - \sigma(g(0))| = \frac{(2+t)(1+|g(0)|)}{6+4t-2|g(0)|}.$$

Let  $\Gamma = \Delta([0, z], P_t; H)$  be the family of curves joining  $[0, z]$  to  $P_t$  in  $H$ , and  $\Gamma' = g(\Gamma) = \Delta(g([0, z]), P_t; H)$ . By the conformal invariance of the modulus, we have

$$M(\Gamma) = M(\sigma(\Gamma)) = \gamma_n \left( \frac{r/2}{|\sigma(z) - \sigma(0)|} \right)$$

and

$$M(\Gamma') = M(\sigma(\Gamma')) \geq \gamma_n \left( \frac{r/2}{|\sigma(z) - \sigma(g(0))|} \right).$$

By  $K$ -quasiconformality we have  $K M(\Gamma) \geq M(\Gamma')$  [V1] implying

$$\frac{1+|g(0)|}{3+2t-|g(0)|} \leq \varphi_{K,n} \left( \frac{1}{3+2t} \right),$$

and further

$$|g(0)| \leq \frac{(3+2t)\varphi_{K,n}(1/(3+2t)) - 1}{1 + \varphi_{K,n}(1/(3+2t))}.$$

Since the above inequality holds for all  $t \geq 0$ , the choice  $t = 0$  gives

$$|g(0)| \leq \frac{3\varphi_{K,n}(1/3) - 1}{1 + \varphi_{K,n}(1/3)}$$

and

$$\rho_{Bn}(0, g(0)) = \log \frac{1 + |g(0)|}{1 - |g(0)|} \leq \log \frac{2\varphi_{K,n}(1/3)}{1 - \varphi_{K,n}(1/3)}.$$

□

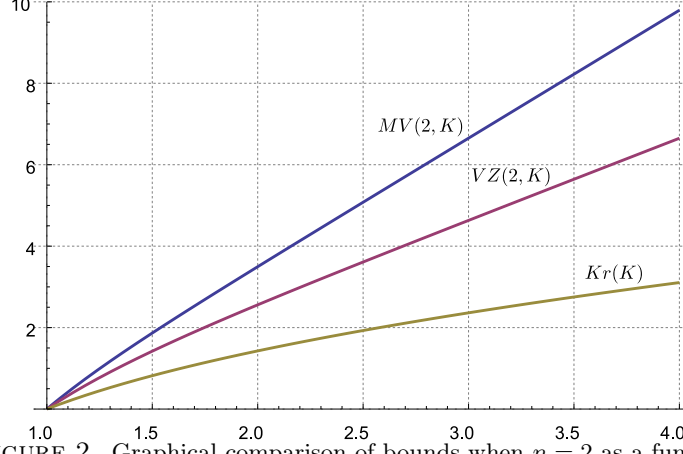


FIGURE 2. Graphical comparison of bounds when  $n = 2$  as a function of  $K$ : (a) the Manojlović and Vuorinen bound

$$MV(2, K) = \log \frac{1 - \varphi_{1/K,2}(1/\sqrt{2})^2}{\varphi_{1/K,2}(1/\sqrt{2})^2},$$

(b) the bounds from Theorem 1.2,

$$VZ(2, K) = \log \frac{2\varphi_{K,2}(1/3)}{1 - \varphi_{K,2}(1/3)},$$

(c) the Krzyż bound only for  $n = 2$

$$Kr(K) = 2 \operatorname{artanh} \mu^{-1} \left( \log \frac{\sqrt{K} + 1}{\sqrt{K} - 1} \right).$$

Note that Vuorinen's example related to (1.3) in Section 1 is unbounded. For bounded domain  $D \subset \mathbb{R}^n$ , however, one can see that for all  $x \in D$ ,  $f(x) \neq x$  implies  $K(f) > 1$  as the following theorem shows.

**3.3. Theorem.** *Let  $D$  be a bounded domain in  $\mathbb{R}^n$ , and  $f \in \operatorname{Id}_K(\partial D)$ . Then for all  $x \in D$*

$$|f(x) - x| \leq \operatorname{diam}(D) \tanh \left( \frac{1}{2} \log \frac{2b}{1-b} \right), \quad b = \varphi_{K,n}(1/3).$$

*Proof.* For  $x \in D$ ,  $D \subset \mathbb{B}^n(x, \operatorname{diam}(D))$  since  $D$  is bounded. Let  $g(w) = (w - x)/\operatorname{diam}(D)$ . It is easy to see that  $h = g \circ f \circ g^{-1} \in \operatorname{Id}_K(\partial \mathbb{B}^n)$ . Hence it follows from Theorem 1.1 that

$$\rho_{\mathbb{B}^n} \left( \frac{f(x) - x}{\operatorname{diam}(D)}, 0 \right) = \rho_{\mathbb{B}^n}(h(0), 0) \leq \log \frac{2b}{1-b},$$

and hence

$$|f(x) - x| \leq \operatorname{diam}(D) \tanh \left( \frac{1}{2} \log \frac{2b}{1-b} \right)$$

since  $\rho_{\mathbb{B}^n}(z, 0) = 2\operatorname{artanh}|z|$  for  $z \in \mathbb{B}^n$ .  $\square$

**3.4. Example.** For sufficiently small  $\epsilon \in (0, 1)$  there exists  $f \in \operatorname{Id}_{1+\epsilon}(\partial(\mathbb{B}^n \setminus \{0\}))$  and  $x_0 \in \mathbb{B}^n \setminus \{0\}$  such that  $k_{\mathbb{B}^n \setminus \{0\}}(x_0, f(x_0)) = 1/\epsilon$ . Actually, we can take  $f$  to be the radial mapping  $f(x) = |x|^{a-1}x$  with  $a = (1 + \epsilon)^{1/(n-1)}$ , and  $x_0 = e^{-b}e_1$  with  $b = 1/(\epsilon(a-1)) < \log 2$ . Then  $K(f) = a^{n-1} = 1 + \epsilon$  (see [V1, 16.2]). It is clear that  $|x_0|, |f(x_0)| < 1/2$ , and hence  $k_{\mathbb{B}^n \setminus \{0\}}(x_0, f(x_0)) = k_{\mathbb{R}^n \setminus \{0\}}(x_0, f(x_0)) = \log(|x_0|/|f(x_0)|) = 1/\epsilon$ .

**3.5. Proof of Theorem 1.4.** Write  $y = f(x)$ . We may assume  $d(x) \leq d(y)$  since  $f^{-1} \in \operatorname{Id}_K(\partial D)$  also. Fix  $z \in \partial D$  such that  $d(x) = |x - z|$ . For  $t > 0$ , write  $F_t = \{z + u(z - x) : u \geq t\}$ . Let  $\Gamma_t = \Delta([x, z], F_t)$  be the family of all curves in  $\mathbb{R}^n$  joining  $[x, z]$  to  $F_t$ .  $\Gamma'_t = f(\Gamma_t) = \Delta(f([x, z]), F_t)$ . From [AVV2, (8.31), Theorem 8.44] it follows that

$$\tau_n \left( \frac{t|x - z|}{|y - z|} \right) \leq M(\Gamma'_t) \leq K M(\Gamma_t) = K \tau_n(t).$$

Setting  $t = 1$ , we have

$$\frac{|y - z|}{|x - z|} \leq \frac{1}{\tau^{-1}(K \tau_n(1))} = \frac{1}{\eta_{1/K, n}(1)},$$

and setting  $t = |y - z|/|x - z|$ ,

$$\frac{|y - z|}{|x - z|} \leq \tau^{-1} \left( \frac{\tau_n(1)}{K} \right) = \eta_{K, n}(1).$$

Hence it follows that

$$\frac{|y - z|}{|x - z|} \leq \min\{\eta_{K, n}(1), \frac{1}{\eta_{1/K, n}(1)}\} = c_2(n, K).$$

Since  $D$  is convex, it is easy to see that  $|y - z|^2 \geq |x - y|^2 + |x - z|^2$ , and hence

$$\frac{|x - y|}{|x - z|} \leq \sqrt{\left( \frac{|y - z|}{|x - z|} \right)^2 - 1}.$$

The definition of the  $j$ -metric, together with the last two inequalities yields

$$j_D(x, y) \leq \log \left( 1 + \sqrt{c_2(n, K)^2 - 1} \right),$$

as desired.  $\square$

In order to prove Theorem 1.5, we need the following two lemmas.

**3.6. Lemma.** *The function*

$$h_1(t) = \frac{\log^2(1+t)}{\log(1+t^2)}$$

*is strictly decreasing in  $(0, 1)$  and strictly increasing in  $(1, \infty)$ . In particular, for  $0 < t \leq 1$ ,*

$$(3.7) \quad \log^2(1+t) \leq \log(1+t^2).$$



*Proof.* Let  $g(t) = (1 + 1/t) \log(1 + t) = g_1(t)/g_2(t)$  with  $g_1(t) = \log(1 + t)$  and  $g_2(t) = t/(1 + t)$ . It is easy to see that  $g_1(0) = 0 = g_2(0)$  and  $g'_1(t)/g'_2(t) = 1 + t$  which is clearly strictly increasing in  $(0, \infty)$ . It follows from the l'Hôpital Monotone Rule [AVV1, Lemma 2.2] that the function  $g$  is strictly increasing in  $(0, \infty)$ .

By elementary computation, we have

$$h'_1(t) = \frac{2t^2 \log(1 + t)}{(1 + t)(1 + t^2) \log^2(1 + t^2)} (g(t^2) - g(t))$$

which is negative for  $t \in (0, 1)$  and positive for  $t \in (1, \infty)$  by the monotonicity of  $g$ . Hence  $h_1$  is strictly decreasing in  $(0, 1)$  and strictly increasing in  $(1, \infty)$ . The inequality (3.7) follows from the monotonicity of  $h_1$  since  $h_1(0+) = 1$ .  $\square$

**3.8. Lemma.** For  $K > 1$ , let  $h_2(t) = t(K^{1/(t-1)} - 1)$ . Then  $h_2$  is strictly decreasing from  $(1, \infty)$  onto  $(\log K, \infty)$ . In particular, for all  $n \geq 2$  and  $K > 1$ ,

$$\log K < n \left( K^{1/(n-1)} - 1 \right) \leq 2(K - 1).$$

*Proof.* Set  $s = K^{1/(t-1)}$ . Then

$$h_2(t) = \frac{s-1}{\log s} \log K + (s-1).$$

It is easy to see that  $s \mapsto (s-1)/\log s$  is strictly increasing on  $(1, \infty)$  and further that  $h_2$  is strictly increasing in  $s$  and hence decreasing in  $t$ .  $\square$

**3.9. Proof of Theorem 1.5.** By (2.5) and (2.6),

$$\eta_{K,n}(1) = \frac{1}{\varphi_{1/K,n}(1/\sqrt{2})^2} - 1 \leq \lambda_n^{2(\beta-1)} 2^\beta - 1.$$

Since  $\lambda_n \leq 2e^{n-1}$  and  $K \leq K_n$ , it follows that

$$\eta_{K,n}(1) \leq (2e^{n-1})^{2(\beta-1)} 2^\beta - 1 \leq 5.$$

Let  $h_1$  be as in Lemma 3.6. Then

$$h_1(\eta_{K,n}(1)^2 - 1) \leq \max\{h_1(0+), h_1(24)\} = h_1(0+) = 1$$

since  $h_1(24) = 0.989 \dots < 1$ . Hence we have

$$\begin{aligned} j_D(x, f(x)) &\leq \log(1 + \sqrt{\eta_{K,n}(1)^2 - 1}) \\ &\leq \sqrt{\log(1 + (\eta_{K,n}(1)^2 - 1))} \\ &\leq \sqrt{2 \log(\lambda_n^{2(\beta-1)} 2^\beta - 1)} \\ &= \sqrt{2 \log(1 + 2(\lambda_n^{2(\beta-1)} 2^{\beta-1} - 1))} \\ &\leq \sqrt{4 \log(1 + \lambda_n^{2(\beta-1)} 2^{\beta-1} - 1)} \\ &= 2\sqrt{\log(2\lambda_n^2)} \sqrt{\beta - 1} \\ &\leq 2\sqrt{2(n-1)} + 2\log 2 \sqrt{\beta - 1} \\ &\leq 2\sqrt{2n} \sqrt{\beta - 1} \\ &\leq 4\sqrt{K-1}, \end{aligned}$$

where the fourth inequality follows from the well-known Bernoulli inequality and the last inequality follows from Lemma 3.8.  $\square$

Since a bounded convex domain is uniform, we have the following estimate for the quasihyperbolic metric.

**3.10. Corollary.** *Let  $D \subsetneq \mathbb{R}^n$  be a bounded convex domain and  $f \in \text{Id}_K(\partial D)$ . Then for all  $x \in D$ .*

$$k_D(x, f(x)) \leq U(D) \log \left( 1 + \sqrt{c_2(n, K)^2 - 1} \right)$$

where  $c_2(n, K)$  is as in Theorem 1.4 and  $U(D)$  is the uniformity constant of the domain  $D$ .

It is well known that the unit ball  $\mathbb{B}^n$  is a uniform domain with the constant  $U(\mathbb{B}^n) = 2$ . This fact, together with Corollary 3.10 and Theorem 1.5, yields the following estimate.

**3.11. Corollary.** *Let  $K \in (1, K_n]$  and  $f \in \text{Id}_K(\partial \mathbb{B}^n)$ , then for all  $x \in \mathbb{B}^n$*

$$k_{\mathbb{B}^n}(x, f(x)) \leq 4\sqrt{2}\sqrt{n(\beta - 1)} \leq 8\sqrt{K - 1}.$$

A slightly modified argument of the proof of Theorem 1.2 can be applied to improve Theorem 1.4 for the case of convex domains as follows. However, it is difficult to obtain an elementary estimate similar to Theorem 1.5 in higher dimensional case ( $n \geq 3$ ) because of lack of efficient estimates for  $1 - \varphi_{K,n}$ .

**3.12. Theorem.** *Let  $D \subsetneq \mathbb{R}^n$  be a convex domain and  $f \in \text{Id}_K(\partial D)$ . Then, for all  $x \in D$ ,*

$$j_D(x, f(x)) \leq \log \left( 1 + \sqrt{\left( \frac{2\varphi_{K,n}(1/3)}{1 - \varphi_{K,n}(1/3)} \right)^2 - 1} \right).$$

*Proof.* We may assume that  $d(x, \partial D) \leq d(f(x), \partial D)$  since  $f^{-1}$  is also in  $\text{Id}_K(\partial D)$ . Let  $z \in \partial D$  with  $d(x, \partial D) = |x - z|$ . For  $t > 0$  let  $P_t$  be the hyperplane perpendicular to  $x - z$  and at distance  $t$  from the point  $z$ , and the half space  $H$  be the component of  $\mathbb{R}^n \setminus P_t$  which contains  $z$ . Let  $\sigma$  be the inversion in the sphere  $S^{n-1}(w, t)$  where  $w = z + 2t(z - x)/|z - x|$ , then we have  $\sigma(H) = \mathbb{B}^n(\sigma(z), t/2)$ . It is easy to see that

$$|\sigma(z) - \sigma(x)| = \left| \frac{t}{2} - \frac{t^2}{2t + |x - z|} \right|$$

and

$$|\sigma(z) - \sigma(y)| = \left| \frac{t}{2} - \frac{t^2}{2t + |f(x) - z|} \right|$$

where  $y = z + |f(x) - z|(x - z)/|x - z|$ .

Let  $\Gamma = \Delta([x, z], P_t; H)$  be the family of curves joining  $[x, z]$  to  $P_t$  in  $H$ , and  $\Gamma' = f(\Gamma) = \Delta(f([x, z]), P_t; H)$ . By the conformal invariance of the modulus and the spherical symmetrization with center at  $z$ ,

$$M(\Gamma) = M(\sigma(\Gamma)) = \gamma_n \left( \frac{t/2}{|\sigma(z) - \sigma(0)|} \right)$$

and

$$M(\Gamma') \geq M(\Delta([y, z], P_t; H)) = \gamma_n \left( \frac{t/2}{|\sigma(z) - \sigma(y)|} \right).$$

By  $K$ -quasiconformality we have  $K M(\Gamma) \geq M(\Gamma')$  implying

$$\frac{|f(x) - z|}{2t + |f(x) - z|} \leq \varphi_{K,n} \left( \frac{|x - z|}{2t + |x - z|} \right).$$

Setting  $2t/|x - z| = (1 - s)/s$ , we have

$$\frac{|f(x) - z|}{|x - z|} \leq \frac{1 - s}{s} \frac{\varphi_{K,n}(s)}{1 - \varphi_{K,n}(s)}.$$

Since  $D$  is convex, it is easy to see that

$$|f(x) - z|^2 \geq |x - f(x)|^2 + |x - z|^2,$$

and hence

$$\frac{|x - f(x)|}{|x - z|} \leq \sqrt{\left( \frac{|f(x) - z|}{|x - z|} \right)^2 - 1}.$$

The definition of the  $j$ -metric, together with the last two inequalities yields

$$j_D(x, f(x)) \leq \log \left( 1 + \sqrt{\left( \frac{1 - s}{s} \frac{\varphi_{K,n}(s)}{1 - \varphi_{K,n}(s)} \right)^2 - 1} \right).$$

Taking  $s = 1/3$ , i.e.  $t = |x - z|$ , we get the inequality as desired.  $\square$

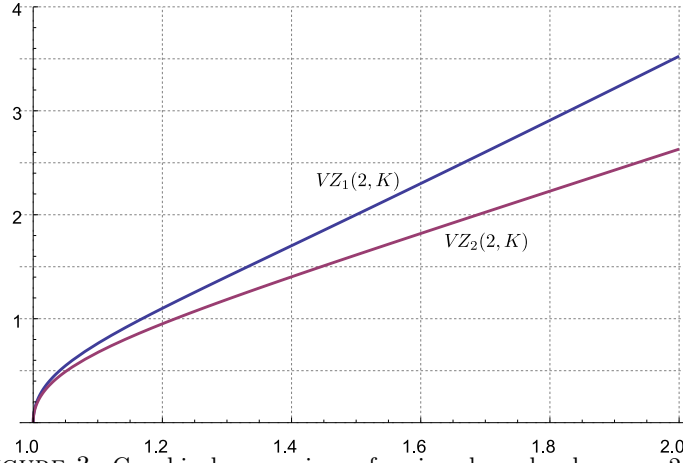


FIGURE 3. Graphical comparison of various bounds when  $n = 2$ , as a function of  $K$ : (a) the bound from Theorem 1.4

$$VZ1(2, K) = \log \left( 1 + \frac{\sqrt{1 - 2\varphi_{1/K,2}(1/\sqrt{2})^2}}{\varphi_{1/K,2}(1/\sqrt{2})^2} \right),$$

(b) the bound from Theorem 3.12:

$$VZ2(2, K) = \log \left( 1 + \sqrt{\left( \frac{2\varphi_{K,n}(1/3)}{1 - \varphi_{K,n}(1/3)} \right)^2 - 1} \right).$$

**3.13. Remark.** For the planar case, by [AVV2, Theorem 10.5 (3)] we have

$$1 - \varphi_{K,2}(r) = (1 + \varphi_{K,2}(r))\varphi_{1/K,2}\left(\frac{1-r}{1+r}\right),$$

which, together with (2.4) and (2.5), yields for  $K > 1$

$$(3.14) \quad 1 - \varphi_{K,2}(r) \geq 4^{1-K}(1+r)^{1-K}(1-r)^K \geq 8^{1-K}(1-r)^K.$$

The inequality (3.14) together with (2.4) gives

$$\log \left( 1 + \sqrt{\left( \frac{2\varphi_{K,n}(1/3)}{1 - \varphi_{K,n}(1/3)} \right)^2 - 1} \right) \sim \sqrt{2 \log 12} \sqrt{K - 1/K}, \quad K \rightarrow 1.$$

**3.15. Open problem.** For  $n \geq 3$ , find a sharp lower bound for  $1 - \varphi_{K,n}(r)$ . Here the sharpness means that the lower bound should tend to  $1 - r$  as  $K \rightarrow 1$ .

**3.16. Proof of Theorem 1.6.** The idea of this proof is exactly the same as in the case of uniform domain with connected boundary. Write  $y = f(x)$ . We may assume  $d(x) \leq d(y)$ . Fix  $z \in \partial D$  such that  $d(x) = |x - z|$ . Then we have  $|y - z| \geq |x - z|$  and

$$(3.17) \quad \frac{|y - z|}{|x - z|} \geq \frac{1}{2} \frac{|x - y|}{d(x)} \geq \frac{1}{2} \left( e^{j_D(x,y)} - 1 \right) \geq \frac{1}{2} \left( e^{k_D(x,y)/U} - 1 \right),$$

where  $U$  is the uniformity constant of the domain  $D$ . Assume now that  $k_D(x, y) \geq 2nU \log(1 + 2e^s)$  where  $s$  is the constant of uniform perfectness of the domain  $D$ . Then this condition together with (3.17) yields

$$(3.18) \quad \frac{|y - z|}{|x - z|} \geq e^{k_D(x,y)/(2U)} \geq (1 + 2e^s)^n.$$

Write  $m = |x - z|$ ,  $M = |y - z|$  and  $t = m^{1/n} M^{1-1/n}$ . Then  $t \in (m, M)$  and  $M/t = (M/m)^{1/n} \geq 1 + 2e^s$ . Let  $[a, x] = \{au + x(1 - u) : 0 \leq u \leq 1\}$ ,  $A = \partial D \setminus \mathbb{B}^n(x, t)$ , and let  $\Gamma_t = \Delta([a, x], A)$  be the family of all curves joining  $[a, x]$  to  $A$ . From Lemma 2.7 and [V1, 7.5] it follows that

$$C \log \frac{M}{t} \leq M(f\Gamma_t) \leq K M(\Gamma_t) \omega_{n-1} \left( \log \frac{t}{m} \right)^{1-n},$$

and hence

$$(3.19) \quad K \geq d_n \left( \log \frac{M}{m} \right)^n; \quad d_n = C(n, s)(n-1)^{n-1}/(\omega_{n-1} n^n),$$

where  $C(n, s)$  depends only on  $s$  and  $n$ , and  $\omega_{n-1}$  depends only on  $n$ . Combining (3.19) and the first inequality of (3.18), we have

$$K \geq d_n (2U)^{-n} k_D(x, y)^n$$

for  $k_D(x, y) \geq 2nU \log(1 + 2e^s)$ . Since  $K \geq 1$ , it is clear that

$$K \geq k_D(x, y)^n / (2nU \log(1 + 2e^s))^n$$

for  $k_D(x, y) < 2nU \log(1 + 2e^s)$ . Hence in all cases

$$K \geq c_3(n, D) k_D(x, y)^n,$$

where  $c_3(n, D) = \min\{d_n(2U)^{-n}, (2nU \log(1 + 2e^s))^{-n}\}$ . □

Next we study the distortion of  $K$ -quasiconformal mappings  $f : \overline{\mathbb{R}}^n \rightarrow \overline{\mathbb{R}}^n$  with the property

$$(3.20) \quad f(te_1) = te_1 \quad \text{for all } t \in \mathbb{R}.$$

The following theorem improves the result of [FV, Theorem 1.6]. Observe that the constant in the theorem tends to one when  $K$  goes to 1.

**3.21. Theorem.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $K$ -quasiconformal mapping which keeps the  $x_1$ -axis pointwise fixed. If  $K > 1$ , then*

$$|f(x)| \leq \left( \lambda_n^{2\beta-2} \frac{\beta^\beta}{(\beta-1)^{\beta-1}} - \lambda^{2-2\beta} \frac{(\beta-1)^{\beta+1}}{4\beta^\beta} \right) |x|$$

for all  $x \in \mathbb{R}^n$  where  $\beta = K^{1/(n-1)}$  and  $\lambda_n$  is the Grötzsch ring constant.

*Proof.* We may assume that  $x \neq 0$  and  $f(x)$  is in the left half space (first coordinate non-positive). For fixed  $s > 0$  let  $h : \overline{\mathbb{R}}^n \rightarrow \overline{\mathbb{R}}^n$  be the Möbius transformation which takes  $f(x)$ ,  $0$ ,  $se_1$ ,  $\infty$  onto  $-e_1$ ,  $-y$ ,  $y$ ,  $e_1$ , respectively, where  $|y| < 1$ . We consider the ring  $R'$  whose complement consists of  $E = h^{-1}([-y, y])$  and  $F = h^{-1}([-e_1, \infty] \cup [e_1, \infty])$ . By [AVV2, Theorem 15.9], we have

$$(3.22) \quad \begin{aligned} \text{cap} R' &\leq \tau_n \left( \frac{|f(x)/s| + |f(x)/s - e_1| - 1}{2} \right) \\ &\leq \tau_n \left( \frac{|f(x)/s| + \sqrt{|f(x)/s|^2 + 1} - 1}{2} \right), \end{aligned}$$

where the second inequality follows from the inequality  $|f(x) - se_1|^2 \geq |f(x)|^2 + s^2$  and the monotonicity of  $\tau_n$ . On the other hand, we put  $R = f^{-1}(R')$  and conclude by [AVV2, Theorem 8.44]

$$(3.23) \quad \text{cap} R \geq \tau_n \left( \frac{|x|}{s} \right).$$

Inequalities (3.22), (3.23), and  $\text{cap} R \leq K \text{cap} R'$  then yield

$$\frac{|f(x)|}{|x|} t + \sqrt{\left( \frac{|f(x)|}{|x|} t \right)^2 + 1} \leq 2\eta_{K,n}(t) + 1, \quad t = \frac{|x|}{s}.$$

Hence

$$(3.24) \quad \begin{aligned} \frac{|f(x)|}{|x|} &\leq \frac{(2\eta_{K,n}(t) + 1)^2 - 1}{2(2\eta_{K,n}(t) + 1)t} \\ &< \frac{(\eta_{K,n}(t) + 1)^2 - 1/4}{(\eta_{K,n}(t) + 1)t} \\ &\leq \lambda_n^{2(\beta-1)} \frac{(1+t)^\beta}{t} - \frac{\lambda^{2(1-\beta)}}{4(1+t)^\beta t}, \end{aligned}$$

where (3.24) follows from the formula (2.6) and the second inequality in (2.5). The choice of  $t = 1/(\beta - 1)$  yields

$$\frac{|f(x)|}{|x|} \leq \lambda_n^{2\beta-2} \frac{\beta^\beta}{(\beta-1)^{\beta-1}} - \lambda^{2-2\beta} \frac{(\beta-1)^{\beta+1}}{4\beta^\beta},$$

and the theorem is proved.  $\square$

3.25. **Proof of Theorem 1.7.** For  $R > 1$  let  $h(x) = x/R$ , then

$$g := h \circ f \circ h^{-1} : \mathbb{B}^n \rightarrow \mathbb{B}^n$$

is a  $K$ -quasiconformal mapping. By applying the following well-known inequality [MRV, 3.1] (also see [Vu2, 3.3])

$$\tanh \frac{\rho(f(x), f(y))}{2} \leq \varphi_{K,n} \left( \tanh \frac{\rho(x, y)}{2} \right)$$

and the estimate for the hyperbolic metric [Vu3, Exercise 2.52(1)]

$$\frac{|x - y|}{1 + |x||y|} \leq \tanh \frac{\rho(x, y)}{2} \leq \frac{|x - y|}{1 - |x||y|}$$

to the mapping  $g$  and points  $x/R, y/R$  for  $x, y \in \mathbb{B}^n$ , we have

$$\frac{|f(x)/R - f(y)/R|}{1 + |f(x)||f(y)|/R^2} \leq \varphi_{K,n} \left( \frac{|x/R - y/R|}{1 - |x||y|/R^2} \right).$$

Hence by (2.4)

$$\begin{aligned} |f(x) - f(y)| &\leq \lambda_n^{1-\alpha} \frac{R + |f(x)||f(y)|/R}{(R - |x||y|/R)^\alpha} |x - y|^\alpha \\ (3.26) \quad &\leq \lambda_n^{1-\alpha} A(R) |x - y|^\alpha, \end{aligned}$$

where

$$A(R) = \frac{R + R^{-1}}{(R - R^{-1})^\alpha}.$$

It is easy to check that  $A(1+) = \infty = A(\infty)$  and

$$R_0 = \sqrt{\frac{1 + \sqrt{\alpha}}{1 - \sqrt{\alpha}}}$$

is the unique value of  $R$  in the interval  $(1, \infty)$  such that  $A'(R) = 0$ . Hence we have

$$C(\alpha) := \min_{1 < R < \infty} A(R) = A(R_0) = 2^{1-\alpha} \alpha^{-\alpha/2} (1 - \alpha)^{(\alpha-1)/2}.$$

Since the inequality (3.26) holds for all  $R > 1$ , we get

$$|f(x) - f(y)| \leq \lambda_n^{1-\alpha} C(\alpha) |x - y|^\alpha.$$

It is easy to see that  $C(1-) = 1$ , and hence  $M_1(n, K) = \lambda_n^{1-\alpha} C(\alpha) \rightarrow 1$  as  $K \rightarrow 1$ .  $\square$

Applying the above theorem to the inverse of  $f$ , we have the following corollary.

3.27. **Corollary.** *If  $f \in \text{Id}_K(\partial \mathbb{B}^n)$ , then for all  $x, y \in \mathbb{B}^n$*

$$\frac{1}{M_2(n, K)} |x - y|^{1/\alpha} \leq |f(x) - f(y)| \leq M_2(n, K) |x - y|^\alpha$$

where  $M_2(n, K) = M_1(n, K)^{1/\alpha}$ .

The following figure shows some upper bounds for Mori's constant, and it should be noted that the first three bounds hold for all quasiconformal self-maps of the unit ball with the origin fixed but without the additional condition of identity boundary values, while the fourth bound holds for all quasiconformal self-maps of the unit ball with identity boundary values but without the condition of the origin fixed.

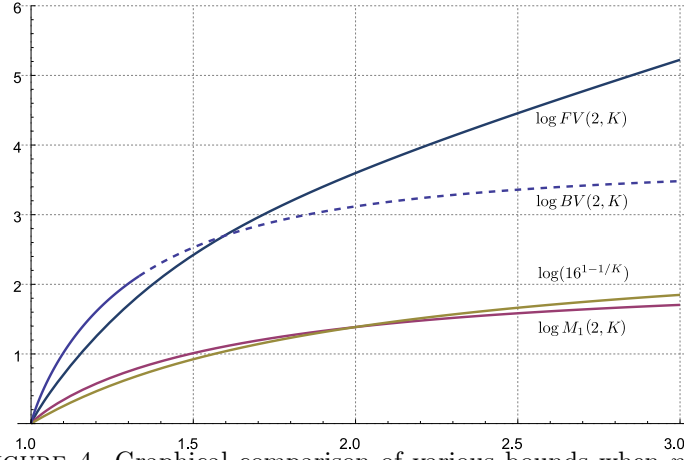


FIGURE 4. Graphical comparison of various bounds when  $n = 2$  and  $\lambda_2 = 4$ , as a function of  $K$ : (a) the Fehlmann and Vuorinen bound [FV]

$$FV(2, K) = \left(1 + \varphi_{2,K} \left( \frac{K^2 - 1}{K^2 + 1} \right)\right) 2^{2K-3/K} \frac{(K^2 + 1)^{(K+1/K)/2}}{(K^2 - 1)^{(K-1/K)/2}},$$

(b) the Bhayo and Vuorinen bound [BV, 1.8, for  $n = 2$ ], valid for  $K \in (1, K_1)$ ,  $K_1 = 4/3$ ,

$$BV(2, K) = 3^{1-1/K^2} 4^{1-1/K-1/K^2} K^2 (K^2 - 1)^{1/K^2-1},$$

(c) Mori's conjectured bound  $16^{1-1/K}$ , (d) the bound  $M_1(2, K)$  from Theorem 1.7.

3.28. **Remark.** In [MV, Remark 3.15], it is proved that

$$\varphi_{K,2}(r) \leq 2\varphi_{K,2} \left( \sqrt{\frac{1+r}{2}} \right)^2 - 1$$

for all  $r \in [0, 1]$ . Writing  $A(r, s) = \sqrt{(r+s)/2}$ , then the authors conjectured that

$$(3.29) \quad A(\varphi_{K,2}(r), \varphi_{K,2}(s)) \leq \varphi_{K,2}(A(r, s))$$

holds for all  $r, s \in [0, 1]$ . Actually, by [WZC]

$$A(\varphi_{K,2}(r), \varphi_{K,2}(s))^2 \leq \varphi_{K,2}(A(r, s)^2),$$

and by [AVV2, Theorem 10.15]

$$\varphi_{K,2}(A(r, s)^2)^{1/2} \leq \varphi_{K,2}(A(r, s)).$$

Now these two inequalities imply (3.29).

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